ANALYSIS OF TWO-VARIABLE FUNCTION GRAPHING ACTIVITIES

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This is a study about the didactical organization of a research based group of activities designed using APOS theory to help university students make constructions, needed to understand and graph two-variable functions, but found to be lacking in previous studies. The model of the "moments of study" of the Anthropological Theory of Didactics is applied to analyze the activities in terms of their institutional viability.

Keywords: Advanced Mathematical Thinking; Geometry and Geometrical and Spatial Thinking

Introduction

Functions of two variables are of great importance in applied mathematics and engineering. However, despite their importance, there are few publications that take advantage of their particularities in order to study their teaching and learning. The first published article we found that explicitly treats functions of two variables is by Yerushalmy (1997). In it she insisted on the importance of the interplay between different representations to generalize key aspects of these functions and to identify changes in what seemed to be fixed properties of each type of function or representation. Kabael (2009) studied the effect that using the "function machine" might have on student understanding of functions of two variables, and concluded that it had a positive impact in their learning. In other work, Montiel, Wilhelmi, Vidakovic, and Elstak (2009) considered student understanding of the relationship between rectangular, cylindrical, and spherical coordinates in a multivariable calculus course. They found that the focus on conversion among representation registers and on individual processes of objectification, conceptualization and meaning contributes to a coherent view of mathematical knowledge. Martínez-Planell and Trigueros (2009) investigated formal aspects of students' understanding of functions of two variables and identified many specific difficulties students have in the transition from one variable to two-variable functions. Using APOS theory, they related these difficulties to specific coordinations that students need to construct among the set, one variable function, and R^3 schemata. In a study about geometric aspects of two variable functions, Trigueros and Martínez-Planell (2010) concluded that students' understanding can be related to the structure of their schema for R^3 and to their flexibility in the use of different representations. These authors gave evidence that the understanding of graphs of functions of two variables is not easy for students and in particular, that intersecting surfaces with planes, and predicting the result of this intersection, plays a fundamental role in understanding graphs of two variable functions and was particularly difficult for students. More concretely, students showed difficulty intersecting fundamental planes (that is, planes of the form $x = c$, $y = c$, or $z = c$ where c is a constant) with surfaces given in different representational formats. Hence they had difficulty with transversal sections, contour curves, and projections. Finally, Trigueros and Martínez-Planell (2011) used the moments of study of the Anthropological Theory of Didactics to analyze the didactical organization of a widely used calculus textbook (Stewart, 2006) and showed that its organization was neither effective in fostering the needed constructions nor viable from the praxeological point of view. This stressed the need to supplement traditional calculus textbooks with activities which cover those aspects found to be lacking in most textbooks. Our research questions in the present study are:

- Does a set of activities designed using a genetic decomposition of functions of two variables help students interiorize actions found to be necessary for a process conception of function of two variables?
- Is the didactical organization of the activity sets conducive to their functioning well at the institutional level?

Theoretical Framework

Since APOS is a well known theory we will only briefly discuss the notion of a genetic decomposition which is important to the content of this paper. For more information on APOS the reader may refer to the brief discussion in Trigueros and Martínez-Planell (2010), or more extensive treatments in Asiala et al (1996), and Dubinsky (1991, 1994).

In APOS Theory, the study of student understanding of a particular concept starts with a "genetic decomposition." This is a hypothesis advanced by the researcher and based on his/ her knowledge, experience, and any available previous data of the actions, processes, and objects that must be constructed in order to attain the desired conceptual understanding. A genetic decomposition is not unique, as different researchers might propose different decompositions. However, it is important that the decomposition be contrasted with data obtained from student interviews to ascertain the constructions actually being made by students. Typically, research data results in revisions of the genetic decomposition as researchers discover unforeseen constructions made by students, or constructions that are assumed to be readily made by students but which are not. The resulting revised genetic decomposition can be used in research and also in the design of activities that may be incorporated into the instructional cycle and that may help students make the desired constructions.

The initial genetic decomposition for function of two variables is given in Trigueros and Martínez-Planell (2010). In order to accommodate results of that study, the genetic decomposition was refined to include the following paragraph on the construction of the schema for R^3, which is important in the present study: The Cartesian plane, real numbers, and the intuitive notion of space schemata must be coordinated in order to construct the Cartesian space of dimension three, R^3, through the action of assigning real numbers to points in R^2, and the actions of representing the results of those actions as 3-tuples, in a table or as points in space. These actions are interiorized into processes that make it possible to consider different sets or subsets, in particular fundamental planes, in each representation register. These processes can be encapsulated into objects on which further treatment actions or processes can be performed. These treatment actions or processes include intersecting fundamental planes with other surfaces to form transversal sections, contour curves and projections, and processes of conversion of those sets and subsets among representations in a schema which evolves and that can be thematized as a schema for three-dimensional space, R^3.

A set of activities was prepared to help students make the constructions suggested by the revised genetic decomposition. We considered it important to analyze and discuss its organization and effectiveness. The *moments of study* of the Anthropological Theory of Didactics (ATD) was used as a tool for the epistemological analysis of the group of activities. In ATD the mathematical activity and the activity of studying mathematics are considered parts of human activity in social institutions (Chevallard, 1997; Bosch & Chevallard, 1999). This theory considers that any human activity can be explained in terms of a system of *praxeologies*, or sets of practices which in the case of mathematical activity constitute the structure of what are called *mathematical organizations* (MO). Mathematical organizations always arise as response to a question or a set of questions. In a specific institution, one or several techniques are introduced to solve a task or a set of tasks. Tasks and the associated techniques, together form what is called the *practical block* of a praxeology. The existence of a technique inside an institution is justified by a technology, where the term "technology" is used in the sense of a discourse or explanation (*logos*) of a technique (*technè*). The technology is justified by a theory. A theory can also be a source of production of new tasks and techniques. Technology and theory constitute the *technological-theoretical block* of a praxeology. Thus a praxeology is a four-tuple $(T/\tau/\theta/\Theta)$ (tasks, techniques, technologies, theories), consisting of a practical block, (T/τ), the tasks and techniques, and a theoretical block, (θ/Θ), made up of the technological and theoretical discourse that explains and justifies the techniques used for the proposed tasks.

Within an educational institution a mathematical praxeology is constructed by a didactic process or a process of study of a MO. This process is described or organized by a model of six moments of study (Chevallard, 2007) which are: *first encounter* with the praxeology, *exploratory moment* to work with tasks

so that techniques suitable for the tasks can emerge and be elaborated, *the technical work moment* to use and improve techniques, the *technological-theoretical moment* where the technological and theoretical discourse takes place, the *institutionalization moment* where the key elements of a praxeology are identified, leaving behind those that only serve a pedagogical purpose, and *evaluation moment* where student learning is assessed and the value of the praxeology is examined. It is important to clarify that the order of the moments is not fixed. It depends on the didactical organization in a given institution, but independently of the order it can be expected that there will be instances where the class will be involved in activities proper to each of the “moments”.

In a recent article, Trigueros, Bosch, and Gascón (2011) discussed the elements of APOS and ATD theories that may be used to expand the theoretical basis of each of these theories without violating their respective basic tenets. They observed that the model of the moments of study may be used in APOS theory to examine instruction based on activities designed in accordance to APOS.

Method

In view of the results obtained in Trigueros and Martínez-Planell (2010, 2011), four activity sets were designed to help students make those constructions found to be needed to understand functions of two variables. The activity sets dealt with (a) fundamental planes and surfaces, (b) cylinders, (c) graphs of functions, (d) contour maps and graphs of functions. All activity sets stress the use of sections in graphical analysis. For example, in a problem of the first activity set students are given the set $S=\left\{(x,y,z): z=x^2+(2+y)^3x+y^2\right\}$ and are asked to draw on a Cartesian plane its intersection with the plane $y=-2$; represent physically the intersection in space (using the manipulative in McGee, 2009); draw in three-dimensional space the resulting intersection curve making sure it is placed in its corresponding plane; and give three points in the intersection. This is to be done right after students are introduced to three-dimensional space, after they have constructed fundamental planes as processes, and before functions of two variables are defined. It aims to have students act on their process of fundamental plane thus helping the encapsulation of fundamental planes into objects. In another problem students are asked to represent physically in space the set $\left\{(x,y,z): z=xy^2, y=0\right\}$ and draw it in three-dimensional space. This is a variation of the algebraic representation of the previous example.

After designing the activity sets, they were analyzed in terms of the genetic decomposition and revised until the researchers agreed they covered those constructions predicted by the genetic decomposition. Then, the moments of study of the ATD were used to analyze their didactic organization in two different institutions. For example, the problems presented above are designed to be part of *the moment of the first encounter*, where students meet an important idea needed to construct their R^3 schema.

Activities were classroom tested and revised in two consecutive semesters. After class testing the activities, a set of interviews was undertaken to evaluate them. This produced new observations leading to further improvements on the activity sets. Fifteen students were chosen and interviewed after they had just finished an undergraduate multivariable calculus course. Of the 15 students, 9 had used the activity sets and 6 had not. Each of these two groups of students had equal number of above average, average, and below average students, as judged by their professors. Each interview lasted approximately 45 minutes. They were transcribed and analyzed independently by the two researchers. The conclusions were negotiated.

The interview questions relevant to this study are reproduced below:

1. Draw or represent in three-dimensional space the set of points in space that satisfy the equation $y=2$ and that are also in the graph of the function $f(x,y)=x^2+y^2$.
2. What can you say of the intersection of the plane $x=0$ with the graph of the function $f(x,y)=x\sin(y)$? Represent the intersection in three-dimensional space.
3. Students were to choose the graph of $f(x,y)=\sin(xy)$ among six given surfaces.

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Results

APOS and Activity Sets

Results suggest that most students who used the activities had an interiorized process of intersecting planes with surfaces. Orlando, who used the activities, obtained a correct graph:

Orlando: I believe this is a cone … it would be… a circle, may I draw it?
Interviewer: Yes, of course
Orlando: … then this is a parabola on the zx plane that is 4 units up… [even though he says "zx" plane he draws and represents it physically correctly in the plane $y = 2$].

Note that even though initially he gave an incorrect answer, Orlando decided the issue by using sections, as practiced repeatedly in the activity sets, thus obtaining the correct graph. The most common student mistakes on the first question were: acting on the familiarity of "$x^2 + y^2$" conclude that the graph was a cylinder (without using sections) and then trying to obtain the intersection geometrically from that graph; and obtaining the correct formula $z = x^2 + 4$ but being convinced this is a parabola on the *xz* plane, not placing it correctly in space. Students not using the activities were more prone to commit these errors as they had less practice intersecting fundamental planes with surfaces and placing the resulting curve in space. Valerie, a student who did not use the activities, seems not to have interiorized the use of sections as a process:

Valerie: … $x^2 + y^2$ would be, a circle … this is harder than I thought … if I draw it … in the *xy* plane, it would be a circle in the *xy* plane, then, if $y = 2$ … it doesn't give the radius…

Question 2 revealed students' difficulties with free variables. Most students did not realize that after substituting $x = 0$ into $z = x\sin(y)$, the variable *y* can take any value, so that the desired intersection is the *y* axis. For example, Jackeline, who had used the activity sets in her class, was able to respond correctly; however it seems she avoids dealing with the free variable by using other sections to visualize the graph of the surface:

Jackeline: … would have the sine function, then as *x* increases the amplitude is going to increase … so this would be a line [under questioning she specifies it is the *y* axis]

On the other hand Victor, also troubled by the free variable, but who did not use the activity sets, does not evidence a process of using sections:

Víctor: $x = 0$, this is confusing … the entire function $x\sin(y)$ becomes 0 … therefore this would be a plane like this and a plane like this… the intersection consists of two planes

In Question 3, the pattern observed in previous questions continued with students who used the activity sets in class showing more of a tendency to use sections and thus performing better.

Activity Sets and the Moments of Study

According to ATD, a balance of the moments of study is needed for materials to help student learning in an institution. As mentioned before, activities that show the importance and usefulness of intersecting fundamental planes with surfaces can be considered as pertaining to the *moment of the first encounter*. The analysis of the activities showed that a large part of them are related to the *moment of task exploration*. This is no wonder, given that in APOS theory reflection on actions so that they may be interiorized into processes, and applying actions to processes so that they may be encapsulated into objects is of fundamental importance, The activity sets start by exploring a wide range of types of tasks aimed at giving students the opportunity to start building a schema for R^3 in which fundamental planes, intersections of fundamental planes with subsets of R^3, free variables, and quadratic surfaces, in different representational formats will be understood. Task exploration continues in the second activity set with cylinders, that is, surfaces in three-dimensional space described with only two variables. This gives the opportunity to have

students reflect on how to plot the graph of $z = x^2$ in three dimensions by initially exploring a point by point representation. In our previous studies we had conjectured that the action of point by point representation may be interiorized into the process of drawing graphs by sections, and this construction was included in the refined genetic decomposition. The last set of interviews showed clearly that this construction is necessary and how a lack of interiorization can act as an obstacle for the coordination of important processes needed to learn the particularities of functions of two variables. The interiorization of actions such as graphing $\{(x,y,0): y=|x|\}$, $\{(x,y,1): y=|x|\}$, and $\{(x,y,2): y=|x|\}$, help students reflect on what is happening as z takes on different values, and can be interiorized when they are asked to draw the graph of $y=|x|$ in three-dimensional space. Coordination of different processes and reflection on them leads students to develop a method for drawing cylinders in three dimensions. Later, in the third and fourth activity sets, tasks explicitly involving the use of those methods for functions of two variables gives students the opportunity to start by point by point construction actions and quickly move on to generalize the constructed processes for other functions like $f(x,y)=x^2+y$. They are also asked to verify their graph by giving values to z and showing the resulting curves as part of the surface drawn previously, an activity which may be considered as part of the *moment of evaluation* as are problems in which students compare their graphs of surfaces to contour diagrams they draw. Other tasks that are explored include darkening the curves where specific fundamental planes intersect a given graph of a surface; matching a given set of formulas to a given set of graphs of surfaces with justifications given in terms of transversal sections, which can be considered as *technological- theoretical moment*. The fourth and final activity set reviews transformations in the context of graphing functions of two variables. The variety of activities in the *moment of task exploration* stresses the use and geometric significance of transversal sections making the *technical work moment* explicit. Many of the problems are broken down into parts to guide students in a step by step construction and reflection on the graphing process. This is in accordance with the didactical approach of APOS theory and is intended to complement traditional textbooks, which (Trigueros & Martinez-Planell, 2011) tend to overlook students' difficulties using transversal sections and contours to graph two-variable functions.

The *technical work moment* is present in the activities as the number and variety of problems enables an increasing number of students to construct a process of graphing functions of two variables with understanding. Activity sets allow students to build a schema for R^3 with the necessary coordinations to sustain ensuing graphing activities. Although traditional books present techniques for graphing functions of two variables, the number and variety of problems directly exploring the use of fundamental planes is limited.

As discussed in Chevallard (2007), the *technological-theoretical moment* is closely interrelated with each of the other moments of study. This is also the case in this topic. The technology of using traces or cross-sections to draw the graph of a two-variable function is introduced in the moment of first encounter and developed with multiple opportunities to do task explorations using the activity sets. Even though the activity sets do not include an explicit discussion of the theory, they include opportunities to discuss and justify the methods used by students; also throughout the activity sets it becomes clear that substituting a number for a variable in an equation with three variables corresponds to intersecting a fundamental plane with the graph of the equation. This being the "*technology*" (in the sense of explanatory discourse) used for graphing functions of two variables, the consistent use of this idea aids the construction of cross-sections, projections, and contours, otherwise found to be difficult for students. Many textbooks typically do not explicitly emphasize the role of fundamental planes in graphing activities and hence students seem to come out of these courses without a clear notion of this "*technology*."

The *moment of institutionalization* is present when the activity sets are formally included in the course syllabus, but more importantly when fundamental planes are explicitly used as an important justification technology throughout the course, for example, when explaining partial derivatives, tangent planes, differential, directional derivatives, iterated integration, and drawing solids whose volumes or mass is to be computed with a double or triple integral. The idea of analyzing a function of two variables by using

knowledge of functions of one variable is pervasive in the course, so that opportunities abound during class discussion for building upon the knowledge of fundamental planes constructed early on in the course, and to institutionalize the processes and objects constructed. Ideas in the activity sets that serve only a pedagogical purpose are not institutionalized; for example, the action of plotting individual points in a graph of a function is quickly interiorized into a process of graphing by sections. The *moment of evaluation* is abundantly available as activity sets present an opportunity for students to auto-evaluate and discuss their work. It is also present if activities are collected and corrected to evaluate students, used in group activities, or used as the basis of test items, or when activity sets themselves are evaluated by studies, such as this one, comparing student performance.

Conclusion

Results suggest that the activity sets help students interiorize actions described in the genetic decomposition of function of two variables into processes, and encapsulate processes into objects and thus, when used effectively, have the potential to improve students' understanding of graphs of functions and their performance in graphing activities. This can only improve as activity sets are iteratively used and discussed in class, refined on the basis of classroom observations, and further studied in depth with successively improved interview instruments, as has been shown in this study. For example, this study uncovered the need to target activities early on that explore the use of free variables, the convenience of using surfaces with graphs that are unlikely to be memorized by students, and the need that some students have of doing a point by point sketch of a graph before they are able to effectively use sections. Some work remains to be done to complete the sets of activities to explore other aspects of the construction of the concept of functions of two variables, such as recognizing domain and range, and working with restricted domains, but so far, interview results show that they improve students' understanding.

The activity sets shows the presence of all the moments required in the study of the graph of these functions. In comparison with traditional texts and courses, the moment of first encounter is clearly present in the activity sets, while the moment of task exploration offers a wide range of activities and opportunities to interiorize actions into processes or to encapsulate processes into objects. The moment of work on the techniques presents the challenge of balancing the number of activities in each set that can realistically be used in class, or be assigned to students. The moment of institutionalization is present when the praxeology developed in the activities is built upon throughout the rest of the course. Finally, the moment of evaluation is present when evaluating student individual and group performance in the activities, including in similar test items, and most importantly, when evaluating the effectiveness of the activity sets per se.

APOS and semiotic representation theories are cognitive theories of learning and as such are limited in their capacity to describe and predict the effects on learning of social and institutional constraints. However, we have shown a situation where one of the models of the ATD can be useful in analyzing the design of activities that result from a cognitive analysis of a learning situation. Constructions and coordinations found to be missing in studies of students' construction of graphs of two-variable functions can be addressed with activities specifically designed to foster those constructions, in a pedagogical organization that takes the different moments of study into account.

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